(040940102/445911B)

M.Sc. DEGREE EXAMINATION, APRIL 2018

FOURTH SEMESTER

Branch – Mathematics

Paper I — GALOIS THEORY

Time : 3 Hours *Max. Marks : 80*

PART – A

Answer any FIVE of the following.

(Marks : 5 × 4 marks = 20 marks)

1. If $f(x) \in F[x]$ is a polynomial of degree 2 or 3. Then show that $f(x)$ is reducible if and only if $f(x)$ has a root in F.

2. If E is a finite extension of F, then show that E is an algebraic extension of F.

3. Construct the splitting fields over Q for the polynomial $x^3 - 1$. Also find the degrees of extension over Q.

4. Show that every finite extension of a finite field is normal.

5. Show that the group $G(Q(\alpha)/Q)$, where $\alpha^5 = 1$ and $\alpha \neq 1$, is isomorophic to the cyclic group of order 4.

6. If F is a field of characteristics $\neq 2$ and $x^2 - a \in F[x]$ is an irreducible polynomial over F. Then show that its Galois group is of order 2.

7. Let $f(x)$ be a polynomial over a field F with no multiple roots. Prove that $f(x)$ is irreducible over F if and only if the Galois group G of $f(x)$ is isomorphic to a transitive permutation group.

8. Show that the polynomial $x^7 - 10x^5 + 15x + 5$ is not solvable by radicals over Q.

PART – B

Answer ONE questions from each unit.

(Marks : 4 × 15 marks = 60 marks)

UNIT – I

9. (a) Let $F \subseteq E \subseteq K$ be fields. If $[K:E] < \infty$ and $[E:F] < \infty$, then show that

   (i) $[K:F] < \infty$

   (ii) $[K:F] = [K:E][E:F]$

   (b) If $p(x)$ is an irreducible polynomial in $F[x]$. Then show that there exists an extension E of F in which $p(x)$ has a root.

Or

[P.T.O]

10. Let E be a extension field of F and let $u \in F$ be algebraic over F. If $p(x) \in F[x]$ polynomial of the least degree such that $p(u) = 0$. Then prove that

(a) $p(x)$ is irreducible over F

(b) If $g(x) \in F[x]$ is such that $g(u) = 0$, then $p(x)/g(x)$.

(c) There is exactly one monic polynomial $p(x) \in F[x]$ of least degree such that $p(u) =$

UNIT – II

11. (a) If $f(x) \in f[x]$ is irreducible over F, then show that all roots of $f(x)$ have the multiplicity.

(b) Show that the prime field of a field F is either isomorphic to Q or to $Z/(p)$, p is p

Or

12. If E is a finite separable extension of a field F, then prove that E is a simple exte of F.

UNIT – III

13. If E is a finite separable extension of a field F, then show the following are equivalent

(a) E is a normal extension of F

(b) F is the fixed field of $G(E/F)$

(c) $[E : F] = |G(E/F)|$.

Or

14. State and prove fundamental theorem of Galois theory.

UNIT – IV

15. (a) If F is a field and U is a finite subgroup of the multiplicative group $F^* = F - \{0\}$ show that U is cyclic.

(b) Show that the Galois group of $x^4 + x^2 + 1$ is the same as that $x^6 - 1$ and is or orde

Or

16. Prove that $f(x) \in F[x]$ is solvable by radicals over F if and only if its splitting field E F has solvable Galois group $G(E/F)$.

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